Name: ……………………………………………………………………………

Gosford High School

Trial HSC 2022

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Section II, show relevant mathematical reasoning and/ or calculations

Total Marks

100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section
- Answer questions on the Multiple-choice answer sheet

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the best response on the multiple-choice answer sheet.

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1. Given $z = \dfrac{1+\sqrt{3}i}{1+i}$, the modulus and argument of the complex number z^5 are respectively

(A) $2\sqrt{2}\,,\dfrac{5\pi}{6}$

(B) $4\sqrt{2}\,,\dfrac{5\pi}{12}$

(C) $4\sqrt{2}\,,\dfrac{7\pi}{12}$

(D) $2\sqrt{2}\,,\dfrac{-\pi}{12}$

2. Which vector represents the unit vector in the direction $\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$?

(A) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (C) $\dfrac{1}{7}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$ (D) $\dfrac{1}{49}\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$

3. The angle between the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} + 5\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{v} = 11\underset{\sim}{i} + 13\underset{\sim}{j} + 17\underset{\sim}{k}$ is closest to

(A) 8°

(B) 11°

(C) 79°

(D) 82°

4. $(\sqrt{3}+i)^n$ is purely real when

(A) $n = 6k\ ,\ k \in \mathbb{Z}$

(B) $n = 6k + \pi\ ,\ k \in \mathbb{Z}$

(C) $n = \dfrac{k}{6}\ ,\ k \in \mathbb{Z}$

(D) $n = \dfrac{6}{k}\ ,\ k \in \mathbb{Z}$

5. Which of the following is the contrapositive of the statement "If Star Wars was released then I did not go school" ?

(A) If I went to school then Star Wars was released.

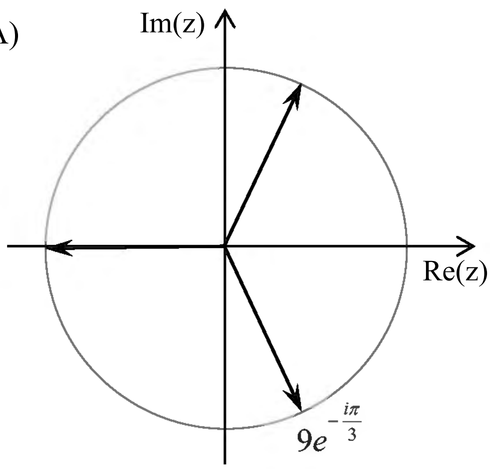
(B) If I did not go to school then Star Wars was not released.

(C) If Star Wars was not released then I went to school.

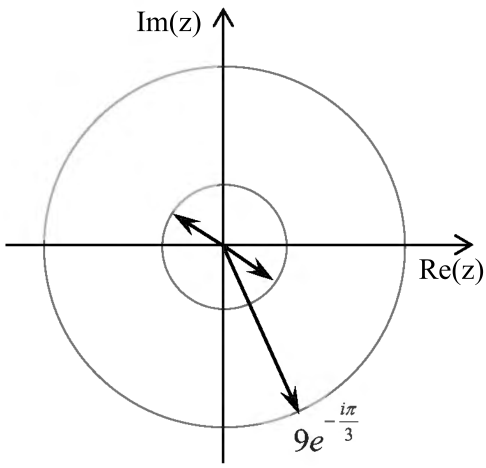
(D) If I went to school then Star Wars was not released.

6. The diagram which best represents the two square roots of $9e^{-\frac{i\pi}{3}}$ is:

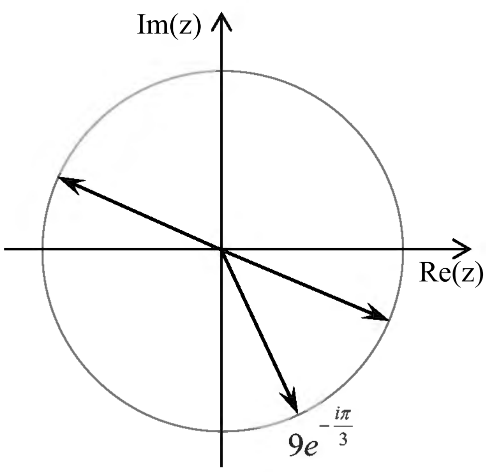
(A)



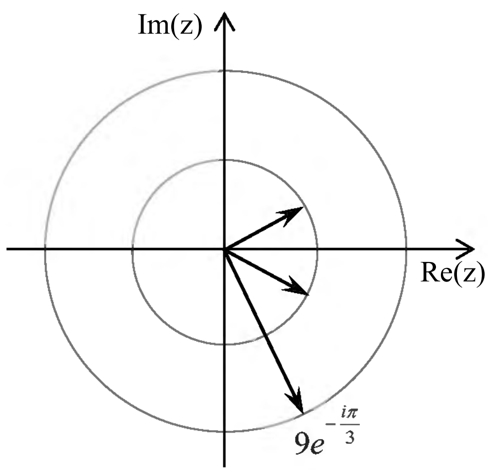
(B)



(C)



(D)



7. Which of the following statements is NOT always true.

(A) $|a+b| \geq |a|-|b|$

(B) $|a-b| \leq |a|-|b|$

(C) $|a+b| \leq |a|+|b|$

(D) $|a-b| \leq |a|+|b|$

8. The polynomial $P(z) = z^4 - 4z^3 + Mz + 20, M \in \mathbb{R}$. Given that $3+i$ is a zero of $P(z)$, which of the following could be a quadratic factor of $P(z)$?

(A) $z^2 + 2z + 2$

(B) $z^2 - 6z + 4$

(C) $z^2 + 6z + 10$

(D) $z^2 - 2z + 20$

9. Which of the following relations has a graph that passes through $1+2i$ on the complex plane?

(A) $\text{Arg(z)} = \dfrac{\pi}{6}$

(B) $|z-1| = |z-2i|$

(C) $\text{Re(z)} = 2\text{Im(z)}$

(D) $z + \overline{z} = 2$

10. Evaluate $\int_0^1 \tan^{-1} x \, dx$

(A) $\dfrac{\pi}{4} - \ln\sqrt{2}$

(B) $\dfrac{\pi}{6} - \ln\sqrt{3}$

(C) $\dfrac{\pi}{2} - \ln\sqrt{5}$

(D) $\dfrac{\pi}{2} - \ln\sqrt{6}$

Section II
90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer in the appropriate booklet provided

Question 11. (15 marks) Answer in the booklet labelled *Question 11*

a) Given that $z_1 = 2 - i$, $z_2 = 1 + 5i$, express the following in the form $a + bi$, where *a, b*, are real.

(i) $\left(\overline{z_1}\right)$ **1**

(ii) $\dfrac{z_1}{z_2}$ **2**

(iii) $\left|z_2 - z_1\right|$ **2**

(iv) $z_1\overline{z_1}$ **2**

b) Prove by counter-example that the proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x = y^2$ is false. **1**

c) Find the cube roots of $8i$ and clearly represent the solutions on an Argand diagram. **3**

d) Let $\overrightarrow{OA} = -\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

(i) Find $\overrightarrow{AB}$. **1**

(ii) Find $|\overrightarrow{AB}|$ **1**

(iii) Find the coordinates of point C such that $\overrightarrow{AC} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ **2**

Question 12 (14 Marks) Answer in the booklet labelled *Question 12*

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a) (i) Show that $\int \frac{2+\ln x}{x^2}dx$ can be expressed as $\int (2+u)e^{-u}\, du$ **2**

(ii) Hence, or otherwise, determine the exact value of $\int_1^e \frac{2+\ln x}{x^2}\, dx$ **3**

b) Determine the vector equation $\underset{\sim}{r}$ of the line segment AB where $A = (1, 2, 1)$ and $B = (4, 10, -3)$. **2**

c) (i) Express $\frac{2x}{(x+1)(x^2+1)}$ in the form $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ where A, B and C are real. **3**

(ii) Hence or otherwise, find $\int_0^1 \frac{2x}{(x+1)(x^2+1)}\, dx$ **4**

Question 13 (14 Marks) Answer in the booklet labelled *Question 13*

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a) Prove by contradiction that for positive real numbers *a, b*, and *c*, if $ab = c$ then $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$ **2**

b) Given that $3 + qi$ is one of the roots of the quadratic equation $z^2 + 12pz + 58 = 0$, where p and q are real constants, find the values of p and q. **3**

c) A sequence is defined by the recursive formula $T_{n+1} = \frac{1}{5}\left(T_n^2 + 6\right)$, $T_1 = \frac{5}{2}$. Prove $T_{n+1} < T_n \ \ \forall\, n > 0, n \in \mathbb{Z}$ by the process of mathematical induction. **3**

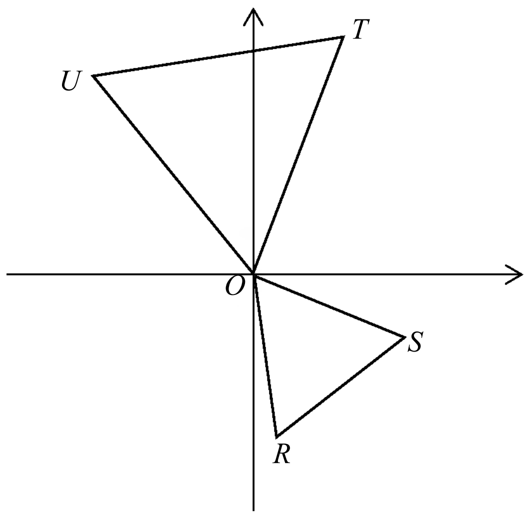
d) Find the cartesian equation of the curve given by $\underset{\sim}{r}(t) = \sqrt{t}\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}$ and hence find the maximum value of the function. **3**

e) Represent on the Argand diagram where the two inequalities $\left|z - i\right| \leq 2$ and $0 \leq \arg(z+1) \leq \frac{\pi}{4}$ both hold. **3**

Question 14 (15 Marks) Answer in the booklet labelled *Question 14*

a) The diagram shows points O, R, S, T and U in the complex plane. These points correspond to the complex numbers 0, r, s, t and u respectively. ΔORS and ΔOTU are equilateral.

Let $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$



(i) Explain why $u = zt$ **1**

(ii) Express the complex number r in terms of s. **2**

(iii) Use complex numbers to show that the lengths of RT and SU are equal. **3**

b) Prove by mathematical induction that $\sum_{k=0}^{n} \frac{1}{3^k} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$ for positive integers n. **3**

c) It is known that $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$. DO NOT PROVE THIS.

If x, y and z are the interior angles of a triangle, show that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{\pi}$ **3**

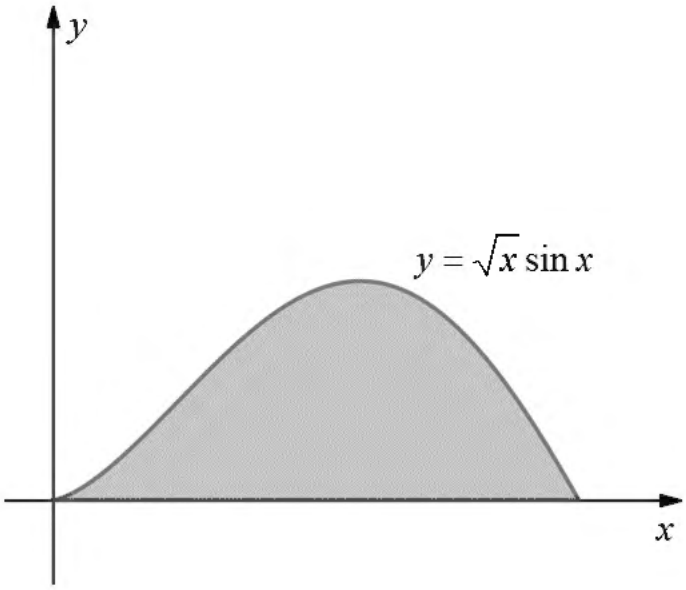
d) Determine the Cartesian equation of the sphere with centre $(9, 20, 12)$ and passes through the point $(25, 0, 0)$. **3**

Question 15 (16 Marks) Answer in the booklet labelled *Question 15*

a) (i) Find $\int \sin^2 x \, dx$ **2**

(ii) Show that $\int x \sin^2 x \; dx = \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$ **3**

(iii) The diagram shows a finite region bound by the curve $y = \sqrt{x}\sin x$ and the x axis.



Find the volume of revolution when the shaded region is rotated about the x axis. **3**

b) Let the roots of $z^3 = -4 + 4\sqrt{3}i$ be z_1, z_2 and z_3.

(i) Find z_1, z_2 and z_3, expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. **3**

(ii) Hence, or otherwise, show that $\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) = 0$ **2**

c) Let $y = x^2 e^x$. Prove by mathematical induction that

$$\frac{d^n}{dx^n} x^2 e^x = (n(n-1) + 2nx + x^2)e^x \quad \forall n \in \mathbb{Z}^+ .$$ **3**

Question 16 (16 Marks) Answer in the booklet labelled *Question 16*

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a) By using the substitution $t = \tan\frac{x}{2}$ or otherwise, find $\int \frac{1}{2+\sin x}\,dx$ **3**

b) The integral I_n is defined by $I_n = \int_0^1 e^x (2-x)^n\,dx$ for integers $n \geq 0$.

(i) Show that $I_n = e - 2^n + nI_{n-1}$ **2**

(ii) Hence or otherwise, evaluate $\int_0^1 e^x (2-x)^4\,dx$ **3**

c) (i) Show that both $\sin^2 x + \cos^4 x$ and $\sin^4 x + \cos^2 x$ can each be expressed as $1 - \frac{1}{4}\sin^2 2x$ **2**

(ii) Show that $\frac{1}{\sin^2 x + \cos^4 x} + \frac{1}{\sin^4 x + \cos^2 x} = \frac{8}{4 - \sin^2 2x}$ **1**

(ii) Hence, or otherwise, show that $2 \leq \frac{1}{\sin^2 x + \cos^4 x} + \frac{1}{\sin^4 x + \cos^2 x} \leq \frac{8}{3}$ **2**

d) Prove that $a^3 + 3a^2 + 2a$, where a is a positive whole number, is divisible by 6 by mathematical induction. **3**

End of Exam

Gosford High School

Trial HSC 2022

Mathematics Extension 2

Sample Solutions and Marking Guidelines

Section I

1	B
2	C
3	A
4	A
5	D
6	B
7	B
8	A
9	D
10	A

Section II

Question 11. (14 marks)

a) Given that $z_1 = 2 - i$, $z_2 = 1 + 5i$, express the following in the form $a + bi$, where *a, b*, are real.

(i) $\left(\overline{z_1}\right)$ **1**

1 mark	Correct answer
Markers Comments	a) Students performed very well in parts (i), (ii), (iii) and (iv), however attention to detail would have eliminated some basic errors when manipulating expressions.

Sample solution

$\overline{z_1} = 2 + i$

(ii) $\dfrac{z_1}{z_2}$ **2**

2 marks	Correct answer
1 mark	Attempt to realise the denominator or equivalent
Markers Comments	See above

Sample solution

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{2-i}{1+5i} \\
&= \frac{2-i}{1+5i} \times \frac{1-5i}{1-5i} \\
&= \frac{2-10i-i+5i^2}{1-25i^2} \\
&= \frac{-3-11i}{26} \\
&= -\frac{1}{26}(3+11i)
\end{aligned}$$

(iii) $|z_2 - z_1|$ **2**

2 marks	Correct answer
1 mark	Evaluates $z_2 - z_1$ correctly or determines modulus of a complex number
Markers Comments	See above

Sample solution

$$\begin{aligned}|z_2 - z_1| &= |(1+5i)-(2-i)| \\ &= |-1+6i| \\ &= \sqrt{(-1)^2+(6)^2} \\ &= \sqrt{37}\end{aligned}$$

(iv) $z_1\overline{z_1}$ **2**

2 marks	Correct answer
1 mark	Progress towards correct answer
Markers Comments	

Sample solution

$$\begin{aligned}z_1\overline{z_1} &= |z_1|^2 \\ &= 2^2+(-1)^2 \\ &= 5\end{aligned}$$

b) Prove by counter-example that the proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x = y^2$ is false. **1**

1 mark	Provides a valid counter-example
Markers Comments	a) Any real negative value for x would have been sufficient to gain the 1 mark.

Sample solution

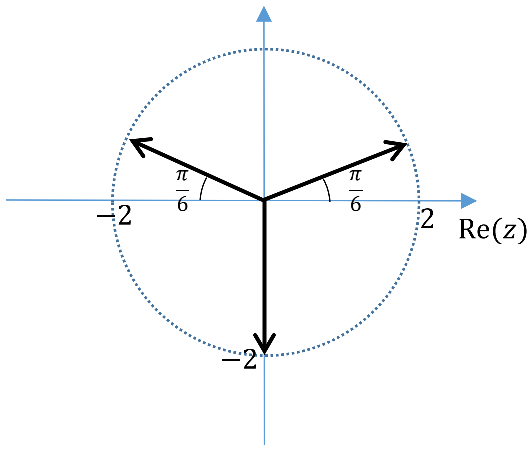
Let $x = -1$. There is no real y such that $y^2 = -1$. Hence the proposition is false.

c) Find the cube roots of $8i$ and clearly represent the solutions on an Argand diagram. **3**

3 marks	Obtains all three correct solutions with representation on the Argand diagram
2 marks	Obtains all three cube roots or represents solutions on the Argand diagram.
1 mark	Obtains one correct cube root or equivalent merit
Markers Comments	a) Generally well done, however students who adopted the following approach will encounter difficulties when dealing with higher powers of z. $z^3 = 8i \Rightarrow z^3 + 8i^3 = 0, \quad \therefore (z+2i)(z^2 - 2iz + 4i^2) = 0\ i.e\ (z+2i)(z^2 - 2iz - 4) = 0$ $Hence\ \ z = 2i\ or\ z = \frac{2i \pm \sqrt{-4+16}}{2}$, leading to $z = -2i\ or\ z = i \pm \sqrt{3}$. Students are reminded to show the roots on a circle of radius 2

Sample solution

$$\begin{aligned}
\text{Let } z^3 &= 8i \\
&= 2^3 \operatorname{cis}\left(\frac{\pi}{2}\right) \\
z_1 &= 2\operatorname{cis}\left(\frac{1}{3}\cdot\frac{\pi}{2}\right) \\
&= 2\operatorname{cis}\left(\frac{\pi}{6}\right) \\
z_2 &= 2\operatorname{cis}\left(\frac{\pi}{6} + \left(\frac{2\pi}{3}\right)\right) \\
&= 2\operatorname{cis}\left(\frac{5\pi}{6}\right) \\
z_3 &= 2\operatorname{cis}\left(\frac{\pi}{6} + 2\left(\frac{2\pi}{3}\right)\right) \\
&= 2\operatorname{cis}\left(\frac{9\pi}{6}\right) \\
&= 2\operatorname{cis}\left(-\frac{\pi}{2}\right)
\end{aligned}$$



$$\therefore z_1 = \sqrt{3} + i,\ z_2 = -\sqrt{3} + i,\ z_3 = -2i$$

d) Let $\overrightarrow{OA} = -\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

(i) Find $\overrightarrow{AB}$. **1**

1 mark	Correct answer
Markers Comments	Most students performed well on this vector question

Sample solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$$
$$= 5\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$$

(ii) Find $|\overrightarrow{AB}|$ **1**

1 mark	Correct answer
Markers Comments	

Sample solution

$$\left|\overrightarrow{AB}\right| = \sqrt{5^2 + 1^2 + (-1)^2}$$
$$= \sqrt{27}$$
$$= 3\sqrt{3}$$

(iii) Find the coordinates of point C such that $\overrightarrow{AC} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ **1**

1 mark	Correct answer
Markers Comments	

Sample solution

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore C\left(-2,1,3\right)$

Question 12 (14 Marks)

a) (i) Show that $\int \frac{2+\ln x}{x^2}\,dx$ can be expressed as $\int (2+u)e^{-u}\,du$ **2**

2 marks	Result logically shown
1 mark	Identifies appropriate substitution and correct derivative.
Markers Comments	a) (i) Show that $\int \frac{2+lnx}{x^2}dx$, required student to show the result step by step rather than jump steps to reach $\int (2+u)\,e^{-u}du$. A few students had difficulty in finding the appropriate expression for dx.

Sample solution

Let $u = \ln x$, hence $x = e^u$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{1}{x}dx$$

$$\int \frac{2+\ln x}{x^2}dx = \int (2+\ln x)\frac{1}{x}\cdot\frac{1}{x}dx$$

$$= \int (2+u)\frac{1}{e^u}du$$

$$= \int (2+u)e^{-u}du$$

(ii) Hence, or otherwise, determine the exact value of $\int_1^e \frac{2+\ln x}{x^2}\,dx$ **3**

3 marks	Correct solution
2 marks	Evaluation of new limits and application of integration by parts, or equivalent merit
1 mark	Evaluation of new limits or application of integration by parts, or equivalent merit
Markers Comments	Well done, however, some students failed to change limits.

Sample solution

$$\int_1^e \frac{2+\ln x}{x^2}\,dx = \int_0^1 (2+u)e^{-u}du$$

$$w = 2+u \qquad v = -e^{-u}$$
$$w' = 1 \qquad v' = e^{-u}$$

$$= \left[-e^{-u}(2+u)\right]_0^1 - \int_0^1 -e^{-u}(1)\,du$$

$$= \left(-\frac{2+1}{e}\right) - \left(-\frac{2+0}{1}\right) - \left[e^{-u}\right]_0^1$$

$$= -\frac{3}{e} + 2 - \left(\frac{1}{e} - 1\right)$$

$$= 3 - \frac{4}{e}$$

b) Determine the vector equation $\underset{\sim}{r}$ of the line segment AB where $A = (1, 2, 1)$ and $B = (4, 10, -3)$. **2**

2 marks	Correct answer, including the restrictions on λ.
1 mark	Identifies vector equation of the line passing through AB or equivalent
Markers Comments	a) Only a few students gained full marks. Many stated that $\lambda = 1$, the required result needed $0 \leq \lambda \leq 1$. Evaluating the values for A, B and C was well done

Sample solution

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4-1 \\ 10-2 \\ -3-1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}$$

But $\underset{\sim}{r}$ represents any point on the line segment AB. When $\lambda = 0$, $\underset{\sim}{r} = \overrightarrow{OA}$. When $\lambda = 1$, $\underset{\sim}{r} = \overrightarrow{OB}$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix} \text{ where } 0 \leq \lambda \leq 1$$

c) (i) Express $\dfrac{2x}{(x+1)(x^2+1)}$ in the form $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$ where A, B and C are real. **3**

3 marks	Correct answer
2 marks	Correct set of simultaneous equations or equivalent
1 mark	Correct expression involving A, B and C
Markers Comments	a) Most students recognised the integrals leading to log and inverse tan however some failed to separate $\frac{x+1}{x^2+1}$ into $\frac{x}{x^2+1}+\frac{1}{x^2+1}$.

Sample solution

$$\frac{2x}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$\therefore A(x^2+1)+(Bx+C)(x+1)=2x$$

Equating coefficients:

$x^2: \quad A+B=0 \quad \boxed{1}$

$x: \quad B+C=2 \quad \boxed{2}$

$c: \quad A+C=0 \quad \boxed{3}$

$\therefore A=-1, \;\; B=1, \;\; C=1$

$$\therefore \frac{2x}{(x+1)(x^2+1)}=-\frac{1}{x+1}+\frac{x+1}{x^2+1}$$

(ii) Hence or otherwise, find $\int_0^1 \frac{2x}{(x+1)(x^2+1)}\, dx$ **4**

4 marks	Correct answer including evaluation	
3 marks	Three correct integration components	As per the third line of the suggested solutions.
2 marks	Two correct integration components	
1 mark	One correct integration	
Markers Comments		

Sample solution

$$\begin{aligned}
\int_0^1 \frac{2x}{(x+1)(x^2+1)}\, dx &= \int_0^1 -\frac{1}{x+1}+\frac{x+1}{x^2+1}\, dx \\
&= \int_0^1 -\frac{1}{x+1}+\frac{1}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{x^2+1}\, dx \\
&= \left[-\ln|x+1|+\frac{1}{2}\ln|x^2+1|+\tan^{-1}x\right]_0^1 \\
&= \left[\ln\left|\frac{\sqrt{x^2+1}}{x+1}\right|+\tan^{-1}x\right]_0^1 \\
&= -\frac{1}{2}\ln 2+\frac{\pi}{4}-\ln 1-0 \\
&= \frac{\pi}{4}-\frac{\ln 2}{2}
\end{aligned}$$

Question 13 (14 Marks) Answer in the booklet labelled *Question 13*

a) Prove by contradiction that for positive real numbers *a, b*, and *c*, if $ab = c$ then $a \le \sqrt{c}$ or $b \le \sqrt{c}$ **2**

2 marks	Valid proof by contradiction
1 mark	Valid negation statement
Markers Comments	Proof by contradiction requires students to firstly negate the statement. Students experienced difficulty negating and “or” statement (see solution). Once you have negated the statement you show there is a contradiction between what is shown and what you are told to be true.

Sample solution

Negation statement: For positive real numbers *a, b*, and *c*, if $ab = c$ then $a > \sqrt{c}$ and $b > \sqrt{c}$.

If $a > \sqrt{c}$ then

$a\sqrt{c} > c$ since $\sqrt{c} > 0$

$\Rightarrow a\sqrt{c} > ab$ since $ab = c$

$\Rightarrow \sqrt{c} > b$

Which contradicts the condition of the negation that $b > \sqrt{c}$

Hence the original statement must be true.

b) Given that $3 + qi$ is one of the roots of the quadratic equation $z^2 + 12pz + 58 = 0$, where p and q are real constants, find the values of p and q. **3**

3 marks	Correct answer
2 marks	Progress towards correct answer
1 mark	Recognition that the roots occur in conjugate pairs or equivalent
Markers Comments	Students should recognise that if a polynomial with integer coefficients then complex roots occur in complex conjuagates.

Sample solution

$3 + qi$ is a root $\Rightarrow 3 - qi$ must also be a root since the coefficients are all real.

Sum of roots: $(3+qi)+(3-qi) = -12p$

$$6 = -12p$$

$$p = -\frac{1}{2}$$

Product of roots: $(3+qi)(3-qi) = 58$

$$9 + q^2 = 58$$

$$q^2 = 49$$

$$q = \pm 7$$

c) A sequence is defined by the recursive formula $T_{n+1} = \frac{1}{5}(T_n{}^2 + 6),\ T_1 = \frac{5}{2}$.

Prove $T_{n+1} < T_n \ \ \forall\, n > 0, n \in \mathbb{Z}$ by the process of mathematical induction. **3**

3 marks	Valid proof by mathematical induction
2 marks	Significant progress towards valid proof
1 mark	Verification of the case for $n = 1$
Markers Comments	Mostly well done. However some students are not writing "by the assumption" when using the induction hypothesis. Other students had trouble applying the recurrence formula.

Sample solution

For $n = 1$, $T_1 = \frac{5}{2}$, $T_2 = \frac{1}{5}\left(\left(\frac{5}{2}\right)^2 + 6\right)$

$$= \frac{49}{20}$$

$$< T_1$$

Assume true for $n = k$. I.e. $T_{k+1} < T_k \ \ \forall\, k > 0, k \in \mathbb{Z}$

R.T.P true for $n = k+1$. I.e. $T_{k+2} < T_{k+1} \ \ \forall\, k > 0, k \in \mathbb{Z}$. Hence, r.t.p. $T_{k+2} - T_{k+1} < 0$

$$LHS = T_{k+2} - T_{k+1}$$

$$= \frac{1}{5}\left((T_{k+1})^2 + 6\right) - \frac{1}{5}\left((T_k)^2 + 6\right)$$

$$\leq \frac{1}{5}\left((T_k)^2 + 6\right) - \frac{1}{5}\left((T_k)^2 + 6\right) \text{ from assumption}$$

$$\leq 0 \text{ as required}$$

Hence true by the process of mathematical induction.

d) Find the cartesian equation of the curve given by $\underset{\sim}{r}(t) = \sqrt{t}\,\underset{\sim}{i} + \frac{1}{t+1}\underset{\sim}{j}$ and hence find the maximum value of the function. **3**

3 marks	Cartesian equation for the function, the restriction and the maximum value.
2 marks	Cartesian equation for the function and either the restriction on domain or the maximum value of the function.
1 mark	Cartesian equation for the function
Markers Comments	Mostly well done, although students should be mindful of the domain.

Sample solution

$$x = \sqrt{t} \quad \Rightarrow t = x^2 \text{ for } x \geq 0$$

$$y = \frac{1}{t+1}$$

$$\therefore y = \frac{1}{x^2+1} \text{ for } x \geq 0$$

The maximum value of $y = \frac{1}{x^2+1}$ occurs at the same x ordinate as the minimum value of $y = x^2 + 1$.

Hence $x = 0$ gives the maximum and minimum values of these respective functions.

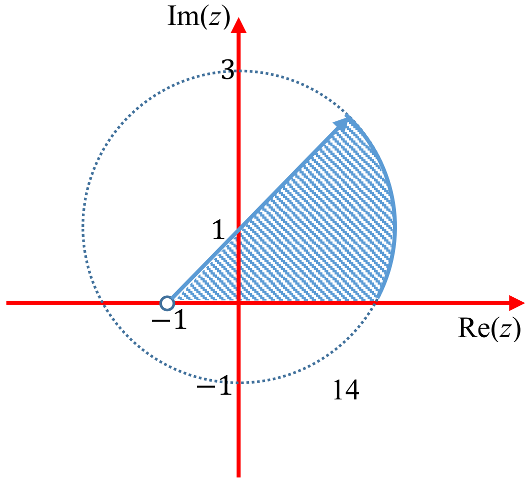
Maximum value of $y = \frac{1}{x^2+1}$ is $y = \frac{1}{0^2+1}$

$$= 1$$

e) Represent on the Argand diagram where the two inequalities $|z - i| \leq 2$ and $0 \leq \arg(z+1) \leq \frac{\pi}{4}$ both hold. **3**

3 marks	Correct diagram
2 marks	Significant progress towards correct diagram
1 mark	One of the inequalities correctly represented on the diagram
Markers Comments	Mostly well done. Please do not draw a solid circle as part of your solution. A solid line means this is part of your solution.

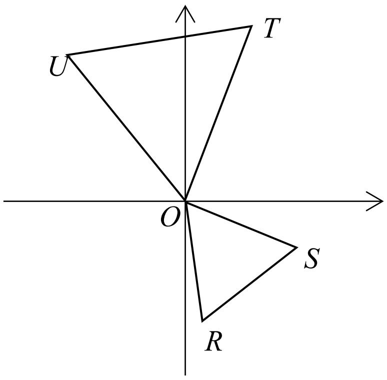
Sample solution



Question 14 (14 Marks)

a) The diagram shows points O, R, S, T and U in the complex plane. These points correspond to the complex numbers 0, r, s, t and u respectively. ΔORS and ΔOTU are equilateral.

Let $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$



(i) Explain why $u = zt$ **1**

1 mark	Correct answer
Markers Comments	Mostly well done, although many students neglected to mention the modulus of z when explaining how complex numbers are rotated.

Sample solution

$|\underset{\sim}{u}| = |\underset{\sim}{t}|$ as they form sides of equilateral ΔUOT and angle $\angle UOT = \frac{\pi}{3}$ (angle in an equilateral triangle).

$|\underset{\sim}{z}| = 1$ and $\text{Arg}(\underset{\sim}{z}) = \frac{\pi}{3}$

$$
\begin{aligned}
\text{Hence } zt &= |z||t|\,cis\left(Arg(z) + Arg(t)\right) \\
&= |t|\,cis\left(Arg(t) + \frac{\pi}{3}\right) \\
&= u
\end{aligned}
$$

As this gives a rotation anticlockwise by $\frac{\pi}{3}$ from t to u.

(ii) Express the complex number r in terms of s. **1**

1 mark	Correct answer
Markers Comments	Mostly well done with some careless errors from some students.

Sample solution

$$r = \frac{s}{z} \text{ or } r = s\overline{z}$$

(iii) Use complex numbers to show that the lengths of RT and SU are equal. **3**

3 marks	Correct solution
2 marks	Valid expressions for both $\left\|\overrightarrow{SU}\right\|$ and $\left\|\overrightarrow{RT}\right\|$ or equivalent merit
1 mark	Valid expression for either $\left\|\overrightarrow{SU}\right\|$ or $\left\|\overrightarrow{RT}\right\|$ or equivalent merit
Markers Comments	Many different solutions to this question and students should be mindful about how they communicate their solution. Neatness and diagrams help.

Sample solution

$$\left|\overrightarrow{SU}\right| = \left|\overrightarrow{SO}\right| + \left|\overrightarrow{OU}\right|$$
$$= -\underset{\sim}{s} + t\underset{\sim}{z}$$
$$\left|\overrightarrow{RT}\right| = \left|\overrightarrow{RO}\right| + \left|\overrightarrow{OT}\right|$$
$$= -\frac{s}{\underset{\sim}{z}} + t$$
$$= \frac{1}{\underset{\sim}{z}}\left(-\underset{\sim}{s} + t\underset{\sim}{z}\right)$$
$$= \overline{\underset{\sim}{z}}\left(-\underset{\sim}{s} + t\underset{\sim}{z}\right)$$
$$= \overline{\underset{\sim}{z}}\left|\overrightarrow{SU}\right|$$

But $\left|\underset{\sim}{z}\right| = 1$ hence $\left|\overrightarrow{RT}\right| = \left|\overrightarrow{SU}\right|$

b) Prove by mathematical induction that $\sum_{k=0}^{n} \frac{1}{3^k} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$ for positive integers n. **3**

3 marks	Valid proof by mathematical induction
2 marks	Significant progress towards valid proof
1 mark	Verification of the case for $n = 1$
Markers Comments	Many students should revise summation induction proofs. Careless errors were made in showing the base case and students need to write "by the assumption" at the induction hypothesis step.

Sample solution

For $n = 1$: $LHS = \frac{1}{3^0} + \frac{1}{3^1} = \frac{4}{3}$ $\quad RHS = \frac{3}{2}\left(1 - \frac{1}{3^2}\right) = \frac{4}{3}$ $\quad$ Hence true for $n = 1$

Assume true for $n = p$. $\sum_{k=0}^{p} \frac{1}{3^k} = \frac{3}{2}\left(1 - \frac{1}{3^{p+1}}\right)$ $\quad$ I.e. $\frac{1}{3^0} + \frac{1}{3^1} + \cdots + \frac{1}{3^p} = \frac{3}{2}\left(1 - \frac{1}{3^{p+1}}\right)$

Required to prove true for $n = p + 1$. $\quad$ I.e. r.t.p $\frac{1}{3^0} + \frac{1}{3^1} + \cdots + \frac{1}{3^p} + \frac{1}{3^{p+1}} = \frac{3}{2}\left(1 - \frac{1}{3^{p+2}}\right)$

$$LHS = \frac{1}{3^0} + \frac{1}{3^1} + \cdots + \frac{1}{3^p} + \frac{1}{3^{p+1}}$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{p+1}}\right) + \frac{1}{3^{p+1}} \text{ from assumption}$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{p+1}} + \frac{2}{3} \cdot \frac{1}{3^{p+1}}\right)$$

$$= \frac{3}{2}\left(1 - \frac{3}{3 \cdot 3^{p+1}} + \frac{2}{3 \cdot 3^{p+1}}\right)$$

$$= \frac{3}{2}\left(1 - \frac{1}{3^{p+2}}\right)$$

$= RHS$ as required

Hence true by the process of mathematical induction.

c) It is known that $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$. DO NOT PROVE THIS.

If x, y and z are the interior angles of a triangle, show that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{\pi}$ **3**

3 marks	Valid proof
2 marks	Application of given inequality to the terms $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and incorporation of $x+y+z=\pi$ or equivalent merit.
1 mark	Application of given inequality to the terms $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ or incorporation of $x+y+z=\pi$ or equivalent merit.
Markers Comments	Students over complicated this question by squaring $x + y + z$. There were a few different methods for this question and students should be mindful to communicate clearly.

Sample solution

$x+y+z=\pi$ (Interior angle sum of a triangle) $\boxed{1}$ Given $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ $\boxed{2}$

Substituting into the given inequality $\boxed{2}$:

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3} \geq \sqrt[3]{\frac{1}{x}\cdot\frac{1}{y}\cdot\frac{1}{z}}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}} \quad \boxed{3}$$

Consider the original inequality $\boxed{2}$:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$

$$\frac{x+y+z}{9} \geq \frac{\sqrt[3]{xyz}}{3}$$

$$\frac{3}{\sqrt[3]{xyz}} \geq \frac{9}{x+y+z}$$

Hence $\frac{3}{\sqrt[3]{xyz}} \geq \frac{9}{\pi}$ $\left(\text{from } \boxed{1}\right)$ $\boxed{4}$

Now applying result $\boxed{4}$ to $\boxed{3}$:

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{\pi}$ as required.

d) Determine the Cartesian equation of the sphere with centre $(9, 20, 12)$ and passes through the point $(25, 0, 0)$. **3**

3 marks	Correct answer in the stated form
2 marks	Further progress towards correct answer
1 mark	Calculation of radius
Markers Comments	Very well done.

Sample solution

$$\begin{aligned}\text{radius} &= \sqrt{(25-9)^2+(0-20)^2+(0-12)^2}\\ &= \sqrt{800}\end{aligned}$$

$$\therefore (x-9)^2+(y-20)^2+(z-12)^2 = 800$$

Question 15 (15 Marks)

a) (i) Find $\int \sin^2 x \, dx$ **1**

1 mark	Correct answer
Markers Comments	Very well done

Sample solution

$$\cos 2x = 1 - 2\sin^2 x \qquad \int \sin^2 x \, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

(ii) Show that $\int x \sin^2 x \; dx = \frac{1}{8}(2x^2 - 2x\sin 2x - \cos 2x) + C$ **3**

3 marks	Correct answer
2 marks	Significant progress towards correct answer
1 mark	Valid substitution into integration by parts formula or equivalent merit
Markers Comments	Very well done

Sample solution

Let $u = x \quad v = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$ from (i)

$u' = 1 \quad v' = \sin^2 x$

$$\int x \sin^2 x \, dx = uv - \int vu' dx$$

$$= \frac{1}{2}x\left(x - \frac{1}{2}\sin 2x\right) - \frac{1}{2}\int\left(x - \frac{1}{2}\sin 2x\right) dx$$

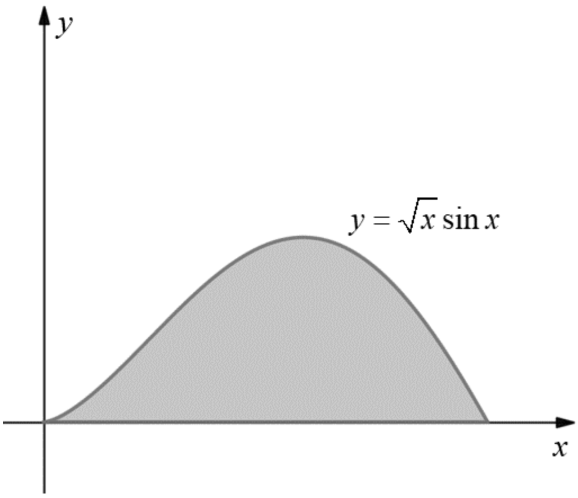
$$= \frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{1}{4}\cos 2x\right) + C$$

$$= \frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{4} - \frac{\cos 2x}{8} + C$$

$$= \frac{1}{8}\left(4x^2 - 2x\sin 2x - 2x^2 - \cos 2x\right) + C$$

$$= \frac{1}{8}\left(2x^2 - 2x\sin 2x - \cos 2x\right) + C$$

(iii) The diagram shows a finite region bound by the curve $y = \sqrt{x}\sin x$ and the x axis.



Find the volume of revolution when the shaded region is rotated about the x axis. **3**

3 marks	Correct answer
2 marks	Further progress towards correct answer
1 mark	Calculation of bounds of integration or equivalent merit
Markers Comments	Very well done

Sample solution

x intercepts: $\sqrt{x}\sin x = 0$

$x = 0, \pi$

Volume of revolution:

$$V = \pi\int_0^{\pi} x\sin^2 x\, dx$$

$$= \frac{\pi}{8}\left[2x^2 - 2x\sin 2x - \cos 2x\right]_0^{\pi}$$

$$= \frac{\pi}{8}\left[\left(2\pi^2 - 2\pi\sin 2\pi - \cos 2\pi\right) - \left(2(0)^2 - 2(0)\sin 2(0) - \cos 2(0)\right)\right]$$

$$= \frac{\pi}{8}\left[2\pi^2 - 0 - 1 - 0 + 0 + 1\right]$$

$$= \frac{\pi^3}{4}$$

b) Let the roots of $z^3 = -4+4\sqrt{3}i$ be z_1, z_2 and z_3.

(i) Find z_1, z_2 and z_3, expressing your answers in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **3**

3 marks	Correct answers expressed in the required form
2 marks	Finds one root and expresses in the form $re^{i\theta}$ or finds three roots but not expressed in the required form, or equivalent merit
1 mark	Express z^3 in the form $re^{i\theta}$ or equivalent merit
Markers Comments	Mostly well done, some students found the modulus incorrectly

Sample solution

$$z^3 = -4+4\sqrt{3}i$$
$$= 8e^{\frac{2\pi}{3}i}$$

Let $z = re^{i\theta}$

$$z_1 = 2e^{\frac{2\pi}{9}i}, \quad z_2 = 2e^{\left(\frac{2\pi}{9}+\frac{2\pi}{3}\right)i}, \quad z_3 = 2e^{\left(\frac{2\pi}{9}+\frac{4\pi}{3}\right)i}$$

$$z_1 = 2e^{\frac{2\pi}{9}i}, \quad z_2 = 2e^{\left(\frac{8\pi}{9}\right)i}, \quad z_3 = 2e^{\left(-\frac{4\pi}{9}\right)i}$$

(ii) Hence, or otherwise, show that $\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=0$ **2**

2 marks	Result shown
1 mark	Valid progress towards solution
Markers Comments	Very well done

Sample solution

Consider the equation:

$$z^3 = -4+4\sqrt{3}i$$
$$z^3+4-4\sqrt{3}i = 0$$
$$\text{I.e. } z^3+0z^2+0z+4-4\sqrt{3}i=0$$

Sum of roots: $z_1+z_2+z_3 = -\frac{b}{a}$ $= 0$

From (i): $2e^{\frac{2\pi}{9}i}+2e^{\left(\frac{8\pi}{9}\right)i}+2e^{\left(-\frac{4\pi}{9}\right)i}=0$

$$e^{\frac{2\pi}{9}i}+e^{\left(\frac{8\pi}{9}\right)i}+e^{\left(-\frac{4\pi}{9}\right)i}=0$$

Hence both the real parts and imaginary parts of the LHS must be 0.

Real part: $\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)+\cos\left(-\frac{4\pi}{9}\right)=0$ But $\cos(-\theta)=\cos(\theta)$

$$\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)=0$$

$$\cos\left(\frac{2\pi}{9}\right)+\cos\left(\frac{4\pi}{9}\right)+\cos\left(\frac{8\pi}{9}\right)=0 \quad \text{as required.}$$

c) Let $y=x^2e^x$. Prove by mathematical induction that

$$\frac{d^n}{dx^n}x^2e^x=(n(n-1)+2nx+x^2)e^x \quad \forall n\in\mathbb{Z}^+.$$ **3**

3 marks	Valid proof by mathematical induction
2 marks	Significant progress towards valid proof
1 mark	Verification of the case for $n=1$
Markers Comments	Mostly well done, some students chose an incorrect value for the initial case

Sample solution

For $n=1$: $LHS=\frac{d}{dx}\left[x^2e^x\right]$ $\qquad$ $RHS=\left(1(1-1)+2(1)x+x^2\right)e^x$

$=x^2e^x+2xe^x$ $\qquad$ $=e^x\left(x^2+2x\right)$

$=e^x\left(x^2+2x\right)$ $\qquad$ $=LHS$ hence true for $n=1$.

Assume true for $n=k$: $\frac{d^k}{dx^k}\left[x^2e^x\right]=(k(k-1)+2kx+x^2)e^x \quad \forall k\in\mathbb{Z}^+$

R.T.P true for $n=k+1$ I.e. r.t.p $\frac{d^{k+1}}{dx^{k+1}}\left[x^2e^x\right]=((k+1)(k)+2(k+1)x+x^2)e^x \quad \forall k\in\mathbb{Z}^+$

$$\begin{aligned}
LHS&=\frac{d^{k+1}}{dx^{k+1}}\left[x^2e^x\right]\\
&=\frac{d}{dx}\left[\frac{d}{dx}\left[x^2e^x\right]\right]\\
&=\frac{d}{dx}\left[(k(k-1)+2kx+x^2)e^x\right] \text{ from assumption}\\
&=(k^2-k+2kx+x^2)e^x+(2k+2x)e^x \text{ by product rule}\\
&=(k^2-k+2kx+2x+2k+x^2)e^x\\
&=(k^2+k+2kx+2x+x^2)e^x\\
&=\left((k+1)(k)+2(k+1)x+x^2\right)e^x\\
&=RHS \text{ as required}
\end{aligned}$$

Hence true by the process of mathematical induction.

Question 16 (16 Marks)

a) By using the substitution $t = \tan\frac{x}{2}$ or otherwise, find $\int \frac{1}{2+\sin x}\,dx$ **3**

3 marks	Correct answer in terms of x
2 marks	Valid integration in terms of t or equivalent merit
1 mark	Valid substitution into integral or equivalent merit
Markers Comments	Well done, however, some students were unable to successfully show that $dx = \frac{2dt}{1=t^2}$

Sample solution

$$\text{Let } t = \tan\frac{x}{2} \qquad \Rightarrow \sin x = \frac{2t}{1+t^2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$$

$$= \frac{1+t^2}{2}$$

$$dx = \frac{2}{1+t^2}dt$$

$$\int\left(\frac{1}{2+\sin x}\right)dx = \int \frac{1}{2+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$= \int \frac{2}{2+2t^2+2t}dt$$

$$= \int \frac{1}{1+t^2+t}dt$$

$$= \int \frac{1}{\frac{3}{4}+\left(t+\frac{1}{2}\right)^2}dt$$

$$= \int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+\left(t+\frac{1}{2}\right)^2}dt$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right)+C$$

b) The integral I_n is defined by $I_n = \int_0^1 e^x (2-x)^n \, dx$ for integers $n \geq 0$.

(i) Show that $I_n = e - 2^n + nI_{n-1}$ **2**

2 marks	Result shown
1 mark	Progress towards the required result
Markers Comments	a) Well done, however, students who incorrectly stated that $\frac{d}{dx}(2-x)^n = n(2-x)^{n-1}$, were either unable to obtain the appropriate Recurrence formula or attempted to fudge the result. (i) Errors when dealing with + and − signs were not uncommon. (ii) Students are again reminded of the importance of attention to detail.

Sample solution

$$\text{Let } u = (2-x)^n \qquad v = e^x$$
$$u' = -n(2-x)^{n-1} \qquad v' = e^x$$

$$\begin{aligned} I_n &= [uv]_0^1 - \int_0^1 vu' \, dx \\ &= \left[e^x (2-x)^n\right]_0^1 - \int_0^1 -e^x n (2-x)^{n-1} \, dx \\ &= e^1 (2-1)^n - e^0 (2-0)^n + n\int_0^1 e^x (2-x)^{n-1} dx \\ &= e - 2^n + nI_{n-1} \end{aligned}$$

(ii) Hence or otherwise, evaluate $\int_0^1 e^x (2-x)^4 \, dx$ **3**

3 marks	Correct answer
2 marks	Significant progress towards correct answer
1 marks	Demonstrates knowledge of how to apply the recursive formula or equivalent merit.
Markers Comments	

Sample solution

Required to find I_4

$$\begin{aligned} I_0 &= \int_0^1 e^x \, dx \\ &= \left[e^x\right]_0^1 \\ &= e - 1 \end{aligned}$$

$$\begin{aligned} I_1 &= e - 2^1 + 1I_0 \\ &= e - 2 + e - 1 \\ &= 2e - 3 \end{aligned}$$

$$\begin{aligned} I_2 &= e - 2^2 + 2I_1 \\ &= e - 4 + 2(2e-3) \\ &= 5e - 10 \end{aligned}$$

$$\begin{aligned} I_3 &= e - 2^3 + 3I_1 \\ &= e - 8 + 3(5e-10) \\ &= 16e - 38 \end{aligned}$$

$$\begin{aligned} I_4 &= e - 2^4 + 4I_1 \\ &= e - 16 + 4(16e-38) \\ &= 65e - 168 \end{aligned}$$

c) (i) Show that both $\sin^2 x+\cos^4 x$ and $\sin^4 x+\cos^2 x$ can each be expressed as $1-\frac{1}{4}\sin^2 2x$ **2**

2 marks	Both results shown
1 mark	Shows one of the required results
Markers Comments	(i) Most students were successful in proving the required result.

Sample solution

$$\begin{aligned}\sin^2 x+\cos^4 x &= \sin^2 x+\cos^2 x\cos^2 x\\ &= \sin^2 x+\cos^2 x(1-\sin^2 x)\\ &= \sin^2 x+\cos^2 x-\cos^2 x\sin^2 x\\ &= 1-(\sin x\cos x)^2\\ &= 1-\left(\frac{1}{2}\sin 2x\right)^2\\ &= 1-\frac{1}{4}\sin^2 2x\end{aligned}$$

$$\begin{aligned}\sin^4 x+\cos^2 x &= \sin^2 x\sin^2 x+\cos^2 x\\ &= \sin^2 x(1-\cos^2 x)+\cos^2 x\\ &= \sin^2 x+\cos^2 x-\sin^2 x\cos^2 x\\ &= 1-(\sin x\cos x)^2\\ &= 1-\left(\frac{1}{2}\sin 2x\right)^2\\ &= 1-\frac{1}{4}\sin^2 2x\end{aligned}$$

(ii) Show that $\frac{1}{\sin^2 x+\cos^4 x}+\frac{1}{\sin^4 x+\cos^2 x}=\frac{8}{4-\sin^2 2x}$ **1**

1 mark	Result logically shown
Markers Comments	(ii) Well done by almost all students. It was pleasing to see that those who were unable to show the result in (i) were able to show the required result in (ii).

Sample solution

$$\begin{aligned}LHS &= \frac{1}{\sin^2 x+\cos^4 x}+\frac{1}{\sin^4 x+\cos^2 x}\\ &= \frac{1}{1-\frac{1}{4}\sin^2 2x}+\frac{1}{1-\frac{1}{4}\sin^2 2x}\\ &= \frac{2}{1-\frac{1}{4}\sin^2 2x}\\ &= \frac{8}{4-\sin^2 2x}\end{aligned}$$

(iii) Hence, or otherwise, show that $2 \le \dfrac{1}{\sin^2 x + \cos^4 x} + \dfrac{1}{\sin^4 x + \cos^2 x} \le \dfrac{8}{3}$ **2**

2 marks	Result logically shown
1 mark	Progress towards the required result
Markers Comments	(iii) Using $-1 \le sinx \le 1 \Longrightarrow 0 \le sin^2x \le 1$ enabled to successfully obtain the minimum and maximum values for the denominator and hence the maximum and minimum values for $\frac{8}{4-sin^2x}$

Sample solution

Consider the range of the known function $4 - \sin^2 2x$

$$0 \le \sin^2 2x \le 1$$

$$3 \le 4 - \sin^2 2x \le 4$$

$$\frac{1}{4} \le \frac{1}{4 - \sin^2 2x} \le \frac{1}{3}$$

$$2 \le \frac{8}{4 - \sin^2 2x} \le \frac{8}{3}$$

Hence $2 \le \dfrac{1}{\sin^2 x + \cos^4 x} + \dfrac{1}{\sin^4 x + \cos^2 x} \le \dfrac{8}{3}$

d) Prove that $a^3 + 3a^2 + 2a$, where a is a positive whole number, is divisible by 6 by mathematical induction. **3**

3 marks	Valid proof by mathematical induction
2 marks	Significant progress towards valid proof
1 mark	Verification of the case for $n = 1$
Markers Comments	(iii) Using $-1 \le sinx \le 1 \Longrightarrow 0 \le sin^2x \le 1$ enabled to successfully obtain the minimum and maximum values for the denominator and hence the maximum and minimum values for $\frac{8}{4-sin^2x}$

Sample solution

For $a = 1$: $1^3 + 3(1)^2 + 2(1) = 6$ which is divisible by six hence true for $a = 1$

Assume true for $a = k$. I.e. $k^3 + 3k^2 + 2k = 6M$, $M \in \mathbb{Z}$.

For $a = k + 1$, r.t.p $(k+1)^3 + 3(k+1)^2 + 2(k+1)$ is divisible by 6.

$$
\begin{aligned}
(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\
&= k^3 + 3k^2 + 2k + 3k^2 + 9k + 6 \\
&= 6M + 3(k^2 + 3k + 2) \\
&= 6M + \underbrace{3\underbrace{(k+2)(k+1)}_{\substack{\text{consecutive integers} \\ \text{hence 2 must be a} \\ \text{factor of one of the} \\ \text{even integer}}}}_{\substack{\textit{Both 3 and 2 are factors} \\ \textit{of this term.}}}
\end{aligned}
$$

$\therefore$ both of these terms are divisible by 6.

Hence true by the process of mathematical induction.

End of Exam